Exercises for Differential calculus in several variables. Bachelor Degree Biomedical Engineering

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Chapter 4.2 Conservative Fields

Problem 1. Consider the force field $\mathbf{F}(x,y,z) = (\sin y + z, x\cos y + \exp^z, x + y\exp^z)$.

i) Show that the path integral along any closed piecewise regular curve equals 0.

ii) Compute a potential function for F, i.e. find a function ϕ such that $\mathbf{F} = \nabla\phi$.

Solution: $i)$ rot $\mathbf{F} = 0$ and $\mathbf{F}$ is of class C^1 on $\mathbb{R}^3$. $ii)$ $\phi(x,y,z) = x(\sin y + z) + ye^z$.

Problem 2. Compute $\int_\gamma \mathbf{F}$, where $\mathbf{F}(x,y,z) = (2xze^{x^2+y^2}, 2yze^{x^2+y^2}, e^{x^2+y^2})$ and γ is the curve in $\mathbb{R}^3$ defined by $\mathbf{r}(t) = (t, t^2, t^3)$, $0 \le t \le 1$.

Solution: $\exp^2$.

Problem 3. Consider the following curve in $\mathbb{R}^3$, $\gamma(t) = \left(e^{t^2} + t(1-e) - 1, \sin^5(\pi t), \cos(t^2 - t)\right)$, $t \in [0,1]$, and the vector field $\mathbf{F}(x,y,z) = (y + z + x^4 \sin x^5,\ x + z + \arctan y,\ x + y + \sin^2 z)$.

i) Compute $\int_\gamma \mathbf{F}$.

ii) Does a function f exist such that $\nabla f = \mathbf{F}$? Find f, if possible.

Solution: $i)$ $\int_\gamma F\cdot dr = 0$ $ii)$ $f(x,y,z) = xy + xz + yz - \frac{1}{5}\cos x^5 + y\arctan y - \log\sqrt{1+y^2} + \frac{1}{2}z - \frac{1}{4}\sin 2z + K$.

Problem 4. Consider the following curve in $\mathbb{R}^3$, $\Gamma = \{\, x^2 + y^2 = 1,\ z = y^2 - x^2 \,\}$, and the vector field $\mathbf{F}(x,y,z) = (y^3, e^y, z)$.

i) Compute $\int_\Gamma \mathbf{F}$.

ii) Does a function f exist such that $\nabla f = \mathbf{F}$?

Solution: $i)$ $-3\pi/4$; $ii)$ No.

Problem 5. Find a and b such that the vector field

$$\mathbf{w}(x,y) = e^{2x+3y}\Big((a\sin x + a\cos y + \cos x),\ (b\sin x + b\cos y - \sin y)\Big)$$

is conservative and compute a potential function in this case.

Solution: $a = 2,\ b = 3;\ \varphi(x,y) = e^{2x+3y}(\sin x + \cos y) + C$.

Problem 6. Consider the vector field

$$\mathbf{F}(x,y) = \Big(\frac{\log(xy)}{x}\,,\ \frac{\log(xy)}{y}\Big),$$

defined for $x > 0$, $y > 0$, and let $a > 0$, $b > 0$ be two constants.

i) Compute $\int_\gamma \mathbf{F}$ where γ is the hyperbola $xy = a$ for $x_1 \le x \le x_2$.

ii) If A is any point on the hyperbola $xy = a$, B is any point on the hyperbola $xy = b$, and γ is any curve of class C^1 contained in the first quadrant that joins A with B, show that

$$\int_\gamma \mathbf{F} = \frac{1}{2}\,\log(ab)\,\log(b/a)\,.$$

Solution: $i)$ $\mathbf{F} = \nabla f$, with $f(x,y) = \frac{1}{2}\big(\log(xy)\big)^2 + c$, $\int_\gamma \mathbf{F} = 0$.